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ON OSCILLATION EQUATION WITH A DELAY

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Introduction

In this paper the solution of the first boundary value problem for a differential oscillation equation with a "pure delay" is investigated. The oscillation equations for the equations without delay were considered by many authors. For instance, various types of oscillation equations were completely investigated in the paper [1]. The functional-differential equations, in particular, with delayed argument, have not been developed so much [2,3]. In a number of papers, the general problems on the existence and uniqueness of solutions were investigated, for special cases of domains; the solutions were obtained in the explicit form. One of the widely used methods for obtaining solutions is the Fourier's method (the method of separation of variables) [2]. In the present paper a scalar equation with a "pure delay" is considered [4]. By using special functions, called delayed cosine and sine [5-7], the solution was obtained in the form of series.

Key words: oscillation equation, delay, boundary value problem, Fourier's method, delayed cosine, delayed sine.

1. Equation without delay. Consider a linear homogeneous differential equation of the form

$$\frac{\partial^2\xi(x,t)}{\partial t^2}=a^2\frac{\partial^2\xi(x,t)}{\partial x^2}+c\xi(x,t) \qquad (1.1)$$

with the given initial and boundary conditions

$$\begin{aligned}&\xi(x,0)=\varphi(x),\ \xi_t'(x,0)=\psi(x),\ 0\le x\le l,\\&\xi(0,t)=\mu_1(t)\ ,\ \xi(l,t)=\mu_2(t),\ 0\le t\le T.\end{aligned} \qquad (1.2)$$

The agreement conditions $\varphi(0)=\mu_1(0)$, $\varphi(l)=\mu_2(l)$ are fulfilled, i.e. at the points $M_1(0,0)$, $M_2(l,0)$ the initial conditions are continuously pass to the boundary ones.

Obtaining of solutions. We preliminarily obtain the solution of the first boundary value problem (1.1), (1.2) for arbitrary $\varphi(x)$, $\psi(x)$, $0\le x\le l$, $\mu_1(t)$, $\mu_2(t)$, $0\le t\le T$. We look for the solution in the form of the sum

$$\xi(x,t)=\xi_0(x,t)+\xi_1(x,t)+\mu_1(t)+\frac{x}{l}[\mu_2(t)-\mu_1(t)], \tag{1.3}$$

- $\xi_0(x,t)$ is the solution of homogeneous equation (1.1) with zero boundary conditions and initial conditions

$$\xi_0(x,0)=\Phi(x),\ \xi_0'(x,0)=\Psi(x),\ \ 0\le x\le l, \tag{1.4}$$

$$\Phi(x)=\varphi(x)-\mu_1(0)-\frac{x}{l}\left[\mu_2(0)-\mu_1(0)\right],$$

$$\Psi(x)=\psi(x)-\mu_1'(0)-\frac{x}{l}\left[\mu_2'(0)-\mu_1'(0)\right],$$

- $\xi_1(x,t)$ is solution of the non-homogeneous equation

$$\frac{\partial^2\xi(x,t)}{\partial t^2}=a^2\frac{\partial^2\xi(x,t)}{\partial x^2}+c\xi(x,t)+F(x,t), \tag{1.5}$$

with the right side

$$F(x,t)=c\left\{\mu_1(t)-\frac{x}{l}[\mu_2(t)-\mu_1(t)]\right\}-\ddot{\mu}_1(t)+\frac{x}{l}[\ddot{\mu}_2(t)-\ddot{\mu}_1(t)], \tag{1.6}$$

and zero boundary and initial conditions.

Homogeneous equation. We'll look for the solution in the form of the product $\xi_0(x,t)=X(x)T(t)$. After substitution into the equation (1.1) we get

$$X(x)T''(t)=a^2X''(x)T(t)+cX(x)T(t).$$

After separating the variables

$$\frac{T''(t)-cT(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\omega^2,$$

the equation decomposes into two equations

$$T''(t)+(a^2\omega^2-c)T(t)=0,\quad X''(x)+\omega^2X(x)=0, \tag{1.7}$$

where ω is some constant.

The solution of the second equation of (1.7), satisfying the zero condition, will be

$$\omega_n=\frac{\pi n}{l},\quad X_n(x)=A_n\sin\frac{\pi n}{l}x,\quad n=1,2,\dots. \tag{1.8}$$

For the choosen ω_n, the first of the equations (1.7) is of the form

$$T_n''(t)+\left(\left(\frac{\pi n}{l}a\right)^2-c\right)T_n(t)=0\,,\quad n=1,2,\ldots. \tag{1.9}$$

Let for definiteness, the parameters a and c be such that

$$k_1^2=\left(\frac{\pi}{l}a\right)^2-c>0\ .$$

Then the solutions of equations (1.9) will be

$$T_n(t)=B_n\cos k_nt+C_n\sin k_nt\,,\quad k_n=\sqrt{\left(\frac{\pi n}{l}a\right)^2-c}\ ,\ n=1,2,\ldots,$$

and the solution of the initial homogeneous equation will be

$$\xi_0(x,t)=\sum_{n=1}^{\infty}[B_n\cos k_nt+C_n\sin k_nt]\sin\frac{\pi n}{l}x\,. \tag{1.10}$$

In order to compute the constants B_n, C_n, $n=1,2,\ldots$ we expand the functions $\Phi(x)$, $\Psi(x)$ a in series

$$\Phi(x)=\sum_{n=1}^{\infty}\Phi_n\sin\frac{\pi n}{l}x\,,\quad \Psi(x)=\sum_{n=1}^{\infty}\Psi_n\sin\frac{\pi n}{l}x\,,\quad n=1,2,\ldots,$$

$$\Phi_n=\frac{2}{l}\int_0^l\Phi(s)\sin\frac{\pi n}{l}sds\,,\quad \Psi_n=\frac{2}{l}\int_0^l\Psi(s)\sin\frac{\pi n}{l}sds\,,\ n=1,2,\ldots. \tag{1.11}$$

Having equated to appropriate values from (1.11), we get

$$B_n=\Phi_n\,,\quad C_nk_n=\Psi_n\,,\quad n=1,2,\ldots..$$

Thus, the solution of the boundary value problem of the homogeneous equation is of the form

$$\xi_0(x,t)=\sum_{n=1}^{\infty}\left[\Phi_n\cos k_nt+\frac{1}{k_n}\Psi_n\sin k_nt\right]\sin\frac{\pi n}{l}x\,,\quad k_n=\sqrt{\left(\frac{\pi n}{l}a\right)^2-c}\,, \tag{1.12}$$

$$\Phi_n=\frac{2}{l}\int_0^l\left\{\varphi(s)-\mu_1(0)-\frac{s}{l}[\mu_2(0)-\mu_1(0)]\right\}\sin\frac{\pi n}{l}sds=$$

$$=\frac{2}{l}\int_0^l\varphi(s)\sin\frac{\pi n}{l}sds-\frac{2}{\pi n}[\mu_1(0)-(-1)^n\mu_2(0)]\,,$$

$$\Psi_n=\frac{2}{l}\int_0^l\left\{\psi(s)-\mu_1'(0)-\frac{s}{l}[\mu_2'(0)-\mu_1'(0)]\right\}\sin\frac{\pi n}{l}sds=$$

$$=\frac{2}{l}\int_0^l\psi(s)\sin\frac{\pi n}{l}sds-\frac{2}{\pi n}[\dot\mu_1(0)-(-1)^n\dot\mu_2(0)]\,. \tag{1.13}$$

Non-homogeneous equation. Let’s consider the first boundary value problem for non-homogeneous equation (1.5)

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = a^2 \frac{\partial^2 \xi(x,t)}{\partial x^2} + c\xi(x,t) + F(x,t),$$

with the function $F(x,t)$ of the form (1.6)

$$F(x,t) = c\left\{\mu_1(t) - \frac{x}{l}[\mu_2(t) - \mu_1(t)]\right\} - \ddot{\mu}_1(t) + \frac{x}{l}[\ddot{\mu}_2(t) - \ddot{\mu}_1(t)],$$

with zero boundary $\xi_1(0,t) = 0$, $\xi_1(l,t) = 0$, $t \geq 0$ and initial $\xi_1(x,0) = 0$, $\xi_1'(x,0) = 0$, $0 \leq x \leq l$ conditions. We'll look for the solution in the form of an expansion in eigenfunctions of the previous problem, i.e. in the form

$$\xi_1(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{\pi n}{l} x. \qquad (1.14)$$

Having substituted the series (1.14) into equation (1.5) and equating the coefficients of the similar terms, we get the system of equations

$$T_n''(t) + \left[\left(\frac{\pi n}{l} a\right)^2 - c\right] T_n(t) = f_n(t),$$

$$f_n(t) = \frac{2}{l}\int_0^l F(s,t) \sin\frac{\pi n}{l} s ds, \; n = 1,2,\ldots. \qquad (1.15)$$

Then the solutions of each of the linear non-homogeneous equations of second-order (1.15) with zero initial conditions will be written in the form

$$T_n^0(t) = \frac{1}{k_n}\int_0^t \sin k_n(t-s) f_n(s) ds, \qquad k_n = \sqrt{\left(\frac{\pi n}{l} a\right)^2 - c}. \qquad (1.16)$$

And the solution of the boundary value problem for the non-homogeneous equation (1.5) with zero boundary condition $\xi(0,t) = 0$, $\xi(1,t) = 0$, $t \geq 0$ and zero initial conditions $\xi(x,0) = 0$, $\xi_t'(x,0) = 0$, $0 \leq x \leq l$ has the form

$$\xi_1(x,t) = \sum_{n=1}^{\infty} \left\{\frac{1}{k_n}\int_0^t \sin k_n(t-s) f_n(s) ds\right\} \sin\frac{\pi n}{l} x, \qquad (1.17)$$

where

$$f_n(t) = -\frac{2}{\pi n}[c\mu_1(t) - \ddot{\mu}_1(t)][(-1)^n - 1] -$$
$$-\frac{2}{\pi n}(-1)^n\{[-c[\mu_2(t) - \mu_1(t)] + [\ddot{\mu}_2(t) - \ddot{\mu}_1(t)]\}. \qquad (1.18)$$

General solution. By using the dependences (1.3), (1.12), (1.17), we write the solution of the first boundary value problem for the equation (1.1) in form of the sum

$$\xi(x,t) = \sum_{n=1}^{\infty} \left[\Phi_n \cos k_n t + \frac{1}{k_n}\Psi_n \sin k_n t + T_n^0(t)\right] \sin\frac{\pi n}{l} x +$$

$$+\sum_{n=1}^{\infty}\left\{\frac{1}{k_n}\int_0^t \sin k_n(t-s) f_n(s)ds\right\}\sin\frac{\pi n}{l}x+\mu_1(t)+\frac{x}{l}[\mu_2(t)-\mu_1(t)]. \quad (1.19)$$

We decompose the last addends in series in eigenfunctions of the Sturm-Liouville problem

$$\mu_1(t)+\frac{x}{l}[\mu_2(t)-\mu_1(t)]=\sum_{i=1}^{\infty}D_n\sin\frac{\pi n}{l}x, \quad (1.20)$$

where

$$D_n=\frac{1}{2l}\int_0^l\left\{\mu_1(t)+\frac{s}{l}[\mu_2(t)-\mu_1(t)]\right\}\sin\frac{\pi n}{l}sds=\frac{2}{\pi n}[(-1)^n\mu_1(t)-\mu_2(t)]. \quad (1.21)$$

Finally the solution of the boundary problem for the equation (1.1) in the form of the three sums that depend on the initial and boundary conditions

$$\xi(x,t)=\sum_{n=1}^{\infty}\left\{\Phi_n\cos k_n t+\frac{1}{k_n}\Psi_n\sin k_n t\right\}\sin\frac{\pi n}{l}x+$$

$$+\sum_{n=1}^{\infty}\left\{\frac{1}{k_n}\int_0^t\sin k_n(t-s)f_n(s)ds\right\}\sin\frac{\pi n}{l}x+$$

$$+\sum_{n=1}^{\infty}\left\{\frac{2}{\pi n}\left[(-1)^n\mu_1(t)-\mu_2(t)\right]\right\}\sin\frac{\pi n}{l}x, \quad (1.22)$$

Where the values Φ_n, Ψ_n, k_n were determined in (1.13).

2. Equation with a pure delay.

Consider a differential equation with one pure delay

$$\frac{\partial^2\xi(x,t)}{\partial t^2}=a^2\frac{\partial^2\xi(x,t-\tau)}{\partial x^2}+c\xi(x,t-\tau), \quad (2.1)$$

with the given initial and boundary conditions

$$\xi(0,t)=\overline{\mu}_1(t),\quad \xi(l,t)=\overline{\mu}_2(t),\ t\ge-\tau,$$
$$\xi(x,t)=\overline{\varphi}(x,t),\quad \xi'(x,t)=\overline{\varphi}'(x,t),\quad 0\le x\le l,\ -\tau\le t\le 0, \quad (2.2)$$

and the “agreement conditions”

$$\overline{\varphi}(0,t)\equiv\mu_1(t),\ \overline{\varphi}(l,t)\equiv\mu_2(t),\ -\tau\le t\le 0,$$

are fulfilled.

We look for the solution in the form of the sum

$$\xi(x,t)=\xi_0(x,t)+\xi_1(x,t)+\overline{\mu}_1(t)+\frac{x}{l}[\overline{\mu}_2(t)-\overline{\mu}_1(t)],$$

where

- $\xi_0(x,t)$ is the solution of homogeneous equation

$$\frac{\partial^2\xi(x,t)}{\partial t^2}=a^2\frac{\partial^2\xi(x,t-\tau)}{\partial x^2}+c\xi(x,t-\tau), \quad (2.3)$$

with zero boundary and non-zero initial conditions

$$\xi_0(x,t)=\Phi(x,t)\,,\,\xi'_{0t}(x,t)=\Phi'_t(x,t)\,,$$

$$\Phi(x,t)=\overline{\varphi}(x,t)-\overline{\mu}_1(t)-\frac{x}{l}[\overline{\mu}_2(t)-\overline{\mu}_1(t)],\ -\tau\le t\le 0,\ 0\le t\le l; \tag{2.4}$$

- $\xi_1(x,t)$ is the solution of non-homogeneous

$$\frac{\partial^2\xi(x,t)}{\partial t^2}=a^2\frac{\partial^2\xi(x,t-\tau)}{\partial x^2}+c\xi(x,t-\tau)+F(x,t), \tag{2.5}$$

$$F(x,t)=c\left\{\overline{\mu}_1(t)+\frac{x}{l}[\overline{\mu}_2(t)-\overline{\mu}_1(t)]\right\}-\ddot{\overline{\mu}}_1(t)-\frac{x}{l}[\ddot{\overline{\mu}}_2(t)-\ddot{\overline{\mu}}_1(t)],$$

with zero boundary and non-zero initial conditions.

Homogeneous equation. We'll look for the solution $\xi_0(x,t)$ of equation (2.2) in the form $\xi_0(x,t)=X(x)T(t)$. After substituting in the equation we get

$$X(x)T''(t)=a^2X''(x)T(t-\tau)+cX(x)T(t-\tau).$$

Separating the variables, we write

$$\frac{T''(t)-cT(t-\tau)}{a^2T(t-\tau)}=\frac{X''(x)}{X(x)}=-\omega^2.$$

We separate the obtained expression into two equations

$$T''(t)+(a^2\omega^2-c)T(t-\tau)=0,\quad X''(x)+\omega^2X(x)=0. \tag{2.6}$$

The solutions of the second equation (2.6), that satisfy zero boundary conditions have the form

$$X_n(x)=A_n\sin\frac{\pi n}{l}x,\quad n=1,2,\ldots.$$

Consider the first of the equations (2.6)

$$T''(t)+k_n^2T(t-\tau)=0,\ k_n=\sqrt{\left(\frac{\pi n}{l}a\right)^2-c},\quad n=1,2,\ldots. \tag{2.7}$$

For obtaining the initial conditions, expand the initial function $\Phi(x,t)$ and its derivative in series by the solutions of the second equation

$$\Phi(x,t)=\sum_{n=1}^{\infty}\Phi_n(t)\sin\frac{\pi n}{l}x,\quad \Phi'_t(x,t)=\sum_{n=1}^{\infty}\Phi'_n(t)\sin\frac{\pi n}{l}x,$$

$$\Phi_n(t)=\frac{2}{l}\int_0^l\overline{\varphi}(s,t)\sin\frac{\pi n}{l}sds-\frac{2}{\pi n}[\overline{\mu}_1(t)-(-1)^n\overline{\mu}_2(t)],\quad n=1,2,\ldots,$$

$$\Phi'_n(t)=\frac{2}{l}\int_0^l\overline{\varphi}'_t(s,t)\sin\frac{\pi n}{l}sds-\frac{2}{\pi n}[\overline{\mu}'_1(t)-(-1)^n\overline{\mu}'_2(t)],\ n=1,2,\ldots. \tag{2.8}$$

We obtain the solution of the Cauchy problem for each of the equations of (2.7) with conditions (2.8).

We beforehand cite some results from the theory of second order differential equations with a pure delay [5,6].

In the paper [6] the following linear homogeneous differential equation of second order and with a constant delay is considered

$$\ddot{x}(t)+\omega^2 x(t-\tau)=0\,,\ \ t\ge 0,\ \ \ \tau>0\,,\ \ x(t)\equiv\varphi(t)\,,\ \ -\tau\le t\le 0\,. \tag{2.9}$$

The functions called delayed cosine and sine are introduced. The solution of the Cauchy problem of differential equation (2.9) is obtained with these functions.

Definition 2.1. The delayed cosine is the function of the form

$$\cos_\tau\{\omega,t\}=\begin{cases} 0, & -\infty<t<-\tau,\\ 1, & -\tau\le t<0,\\ 1-\omega^2\dfrac{t^2}{2!}, & 0\le t<\tau,\\ \dots\dots\dots\dots & \dots\dots\\ 1-\omega^2\dfrac{t^2}{2!}+\dots+(-1)^k\omega^{2k}\dfrac{[t-(k-1)\tau]^{2k}}{(2k)!}, & (k-1)\tau\le t<k\tau \end{cases} \tag{2.10}$$

polynomials of degree $2k$ on the closed intervals $(k-1)\tau\le t\le k\tau$, attached at the nodes $t=k\tau$, $k=0,1,2,\dots$.

Definition 2.2. Delayed sine is called a function of the form

$$\sin_\tau\{\omega,t\}=\begin{cases} 0, & -\infty<t<-\tau,\\ \omega(t+\tau), & -\tau\le t<0,\\ \omega(t+\tau)-\omega^3\dfrac{t^3}{3!}, & 0\le t<\tau,\\ \dots\dots\dots\dots & \dots\dots\\ \omega(t+\tau)-\omega^3\dfrac{t^3}{3!}+\dots+(-1)^k\omega^{2k+1}\dfrac{[t-(k-1)\tau]^{2k+1}}{(2k+1)!}, & (k-1)\tau\le t<k\tau \end{cases} \tag{2.11}$$

polynomials of degree $(2k+1)$ on the closed intervals $(k-1)\tau\le t\le k\tau$, attached at the nodes $t=k\tau$, $k=0,1,2,\dots$.

It was proved that the delayed cosine is the solution of second order differential equation with pure delay (2.9), satisfying the unit initial condition $x(t)\equiv 1$, $-\tau\le t\le 0$, and the delayed sine is the solution of the equation with delay (2.9), satisfying the initial condition $x(t)\equiv\omega(t+\tau)$, $-\tau\le t\le 0$.

By using these facts, the solution of the Cauchy problem is obtained in the compact form. More exactly, the solution $x(t)$ of homogeneous differential equation with delay (2.9), satisfying the initial conditions $x(t)\equiv\varphi(t)$, $x'(t)\equiv\varphi'(t)$, $-\tau\le t\le 0$ where $\varphi(t)$ is an arbitrary twice continuous differentiable function, is of the form [6]

$$x(t)=\varphi(-\tau)\cos_\tau\{\omega,t\}+\frac{1}{\omega}\dot{\varphi}(-\tau)\sin_\tau\{\omega,t\}+$$
$$+\frac{1}{\omega}\int_{-\tau}^{0}\sin_\tau\{\omega,t-\tau-\xi\}\ddot{\varphi}(\xi)d\xi\,. \tag{2.12}$$

For the non-homogeneous differential equation

$$\ddot{x}(t)+\omega^2 x(t-\tau)=f(t),\ t\geq 0,\ \tau>0. \tag{2.13}$$

The solution $x_0(t)$, satisfying the zero initial condition $x(t)\equiv 0$, $-\tau\leq t\leq 0$, has the form [6]

$$x_0(t)=\int_0^t \sin_\tau\{\omega,\ t-\tau-\xi\}f(\xi)d\xi. \tag{2.14}$$

Combining the obtained results, we get that the solution $x(t)$ of the non-homogeneous differential equation with delay (2.13) satisfying the initial conditions $x(t)\equiv\varphi(t)$, $x'(t)\equiv\varphi'(t)$, $-\tau\leq t\leq 0$ where $\varphi(t)$ is an arbitrary twice continuous differentiable function, is of the form

$$x(t)=\varphi(-\tau)\cos_\tau\{\omega,t\}+\frac{1}{\omega}\dot{\varphi}(-\tau)\sin_\tau\{\omega,t\}\omega t+$$

$$+\frac{1}{\omega}\int_{-\tau}^{0}\sin_\tau\{\omega,t-\tau-\xi\}\ddot{\varphi}(\xi)d\xi+\int_0^t \sin_\tau\{\omega,t-\tau-\xi\}f(\xi)d\xi. \tag{2.15}$$

Returning to the partial differential equation, we find that the solution $\xi_0(x,t)$ of the homogeneous equation (2.3) that satisfies the zero and non-zero initial conditions $\xi(x,t)=\Phi(x,t)$, $-\tau\leq t\leq 0$, $0\leq x\leq l$ has the form

$$\xi_0(x,t)=\sum_{n=1}^{\infty}\left\{\Phi_n(-\tau)\cos_\tau\{k_n,t\}+\frac{1}{k_n}\Phi'_n(-\tau)\sin_\tau\{k_n,t\}+\right.$$

$$\left.+\frac{1}{k_n}\int_{-\tau}^{0}\sin_\tau\{k_n,t-\tau-\xi\}\Phi''_n(\xi)d\xi\right\}\sin\frac{\pi n}{l}x,$$

$$k_n=\sqrt{\left(\frac{\pi n}{l}a\right)^2-c},\quad n=1,2,\ldots. \tag{2.16}$$

$$\Phi'_n(t)=\frac{2}{l}\int_0^l \bar{\varphi}'_t(s,t)\sin\frac{\pi n}{l}s\,ds-\frac{2}{\pi n}[\bar{\mu}'_1(t)-(-1)^n\bar{\mu}'_2(t)],\quad n=1,2,\ldots,$$

$$\Phi_n(t)=\frac{2}{l}\int_0^l \bar{\varphi}(s,t)\sin\frac{\pi n}{l}s\,ds+\frac{2}{\pi n}[(-1)^n\bar{\mu}_2(t)-\bar{\mu}_1(t)],\quad n=1,2,\ldots.$$

Non-homogeneous equation. Consider the non-homogeneous equation

$$\frac{\partial^2\xi(x,t)}{\partial t^2}=a^2\frac{\partial^2\xi(x,t-\tau)}{\partial x^2}+c\xi(x,t-\tau)+F(x,t), \tag{2.17}$$

$$F(x,t)=c\left\{\bar{\mu}_1(t-\tau)+\frac{x}{l}[\bar{\mu}_2(t-\tau)-\bar{\mu}_1(t-\tau)]\right\}-\ddot{\bar{\mu}}_1(t)-\frac{x}{l}[\ddot{\bar{\mu}}_2(t)-\ddot{\bar{\mu}}_1(t)],$$

with zero boundary and initial conditions. We looking for the solution in the form

$$\xi_1(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\frac{\pi n}{l}x\,. \qquad (2.18)$$

Substituting (2.18) into equation (2.17) and equating the appropriate coefficients, we obtain the system of second order equations with delay

$$T_n''(t)+\left[\left(\frac{\pi n}{l}a\right)^2-c\right]T_n(t-\tau)=f_n(t)\,, \qquad (2.19)$$

$$f_n(t)=\frac{2}{l}\int_0^l F(s,t)\sin\frac{\pi n}{l}sds=\frac{2}{l}\int_0^l\left\{c\left[\overline{\mu}_1(t-\tau)+\frac{s}{l}[\overline{\mu}_2(t-\tau)-\overline{\mu}_1(t-\tau)]\right]-\right.$$

$$\left.-\overline{\mu}_1''(t)-\frac{s}{l}[\overline{\mu}_2''(t)-\overline{\mu}_1''(t)]\sin\frac{\pi n}{l}sds\right\}.$$

By using the relation (2.14), we write the solution of each of equations of (2.19), satisfying the zero initial conditions, in the form

$$T_n(t)=\int_0^t \sin_\tau\{k_n,t-\tau-s\}f_n(s)ds\,,\ \ k_n=\sqrt{\left(\frac{\pi n}{l}a\right)^2-c}\,,\quad n=1,2,\ldots \qquad (2.20)$$

And the solution of the non-homogeneous equation with zero boundary and zero initial conditions accordingly, will have the form

$$\xi_1(x,t)=\sum_{n=1}^{\infty}\left\{\int_0^t \sin_\tau\{k_n,t-\tau-s\}f_n(s)ds\right\}\sin\frac{\pi n}{l}x\,,$$

$$f_n(t)=-\frac{2}{\pi n}[c\overline{\mu}_1(t-\tau)-\overline{\mu}_1''(t)][(-1)^n-1]-$$

$$-\frac{2}{\pi n}\{c[\overline{\mu}_2(t-\tau)-\overline{\mu}_1(t-\tau)]-[\overline{\mu}_2''(t)-\overline{\mu}_1''(t)]\}\,. \qquad (2.21)$$

General solution. The general solution of the first boundary value problem has the form of the sum

$$\xi(x,t)=\sum_{n=1}^{\infty}\left\{\Phi_n(-\tau)\cos_\tau\{k_n,t\}+\frac{1}{k_n}\Phi_n'(-\tau)\sin_\tau\{k_n,t\}+\right.$$

$$\left.+\frac{1}{k_n}\int_{-\tau}^{0}\sin_\tau\{k_n,t-\tau-\xi\}\Phi_n''(\xi)d\xi+\int_0^t\sin_\tau\{k_n,t-\tau-s\}f_n(s)ds\right\}\sin\frac{\pi n}{l}x+$$

$$+\mu_1(t)+\frac{x}{l}[\mu_2(t)-\mu_1(t)]\,. \qquad (2.22)$$

Expanding the last addend in series, we get

$$\xi(x,t)=\sum_{n=1}^{\infty}\left\{\Phi_n(-\tau)\cos_\tau\{k_n,t\}+\frac{1}{k_n}\Phi_n'(-\tau)\sin_\tau\{k_n,t\}+\right.$$

$$+\frac{1}{k_n}\int_{-\tau}^{0}\sin_\tau\{k_n,t-\tau-\xi\}\Phi_n''(\xi)d\xi+\int_0^t\sin_\tau\{k_n,t-\tau-\xi\}f_n(\xi)d\xi+$$

$$+\frac{2}{\pi n}[(-1)^n\bar{\mu}_1(t)-\bar{\mu}_2(t)]\Big\}\sin\frac{\pi n}{l}x,\ k_n=\sqrt{\left(\frac{\pi n}{l}a\right)^2-c},\ n=1,2,\dots \quad (2.23)$$

$$\Phi_n(t)=\frac{2}{l}\int_0^l\bar{\varphi}(s,t)\sin\frac{\pi n}{l}s\,ds-\frac{2}{\pi n}[\bar{\mu}_1(t)-(-1)^n\bar{\mu}_2(t)],$$

$$f_n(t)=-\frac{2}{\pi n}[c\bar{\mu}_1(t-\tau)-\bar{\mu}_1''(t)][(-1)^n-1]-$$

$$-\frac{2}{\pi n}\{c[\bar{\mu}_2(t-\tau)-\bar{\mu}_1(t-\tau)]-[\bar{\mu}_2''(t)-\bar{\mu}_1''(t)]\}.$$

Convergence condition. The represented dependence (2.23) has the form of the Fourier series. Having imposed the condition of convergence of the series, we formulate the following theorem on the solution of the boundary value problem of the equation with delay.

Theorem 2.1. Let the function $\bar{\varphi}(x,t)$, $\bar{\mu}_1(t)$ and $\bar{\mu}_2(t)$ be such that for $\Phi_n(t)$ and $F_n(t)$, that were determined in (2.8), (2.21), the following conditions are fulfilled:

$$\lim_{n\to\infty}\max_{0\le t\le t^*}|F_n(t)|n^{2k+3+\alpha}=0,\quad \lim_{n\to\infty}|\Phi_n'(-\tau)|n^{2k+3+\alpha}=0,$$

$$\lim_{n\to\infty}|\Phi_n''(-\tau)|n^{2k+3+\alpha}=0,\ \alpha>0,\ -\tau\le t\le T-\tau,\ (k-1)\tau\le T<k\tau, \quad (2.24)$$

Then the solution of the problem (2.1), (2.2) has the form (2.22) (or (2.23)). Therefore, the double differentiation with respect to x and with respect to t, is possible, and the obtained series converge absolutely and uniformly for $0\le x\le l$, $0\le t<T$.

Proof. Make the following transformation. Write (2.22) in the form of sum

$$\xi(x,t)=S_1(x,t)+S_2(x,t)+S_3(x,t)+\bar{\mu}_1(x,t)+\frac{x}{l}[\bar{\mu}_2(t)-\bar{\mu}_1(t)],$$

where

$$S_1(x,t)=\sum_{n=1}^{\infty}A_n(t)\sin\frac{\pi n}{l}x,$$

$$S_2(x,t)=\sum_{n=1}^{\infty}B_n(t)\sin\frac{\pi n}{l}x,$$

$$S_3(x,t)=\sum_{n=1}^{\infty}C_n(t)\sin\frac{\pi n}{l}x,$$

$$A_n(t)=\Phi_n(-\tau)\cos_\tau\{k_n,t\}+\frac{l}{\pi n\sqrt{\lambda_1}}\Phi_n'(-\tau)\sin_\tau\{k_n,t\},$$

$$B_n(t)=\frac{l}{\pi n\sqrt{\lambda_1}}\int_{-\tau}^{0}\sin_\tau\{k_n,t-\tau-s\}\Phi_n''(s)ds,$$

$$C_n(t)=\frac{l}{\pi n\sqrt{\lambda_1}}\int_{-\tau}^{0}\sin_\tau\{k_n,t-\tau-s\}f_n(s)ds,\ k_n=\sqrt{\left(\frac{\pi n}{l}a\right)^2-c}.$$

1. Consider the coefficients $A_n(t)$. For an arbitrary $T:\ (k-1)\tau\le T<k\tau$ there will be fulfilled

$$A_n(t)=\Phi_n(-\tau)\cos_\tau\{k_n,T\}+\frac{l}{k_n}\Phi_n'(-\tau)\sin_\tau\{k_n,T\}=$$

$$=\Phi_n(-\tau)\left\{1-\left(\left(\frac{\pi n}{l}a\right)^2-c\right)\frac{(T-\tau)^2}{2!}+\ldots\right.$$

$$\left.\ldots+(-1)^k\left(\left(\frac{\pi n}{l}a\right)^2-c\right)\frac{[T-(k-1)\tau]^{2k}}{(2k)!}\right\}+$$

$$+\Phi_n'(-\tau)\left\{(T+\tau)-\left(\left(\frac{\pi n}{l}a\right)^2-c\right)^{\frac{3}{2}}\frac{T^3}{3!}+\ldots\right.$$

$$\left.\ldots+(-1)^k\left(\left(\frac{\pi n}{l}a\right)^2-c\right)^{\frac{1}{2}(2k+1)}\frac{[T-(k-1)\tau]^{2k+1}}{(2k+1)!}\right\}.$$

And if the coefficients $\Phi_n(-\tau)$ and $\Phi_n'(-\tau)$ are such that the condition

$$\lim_{n\to+\infty}|\Phi_n(-\tau)|n^{2k+2+\alpha}=0,\ \lim_{n\to+\infty}|\Phi_n'(-\tau)|n^{2k+3+\alpha}=0,$$

is fulfilled, the the series $S_1(x,T)$ converges absolutely and uniformly.

2. Consider the coefficients $B_n(t)$. Make the substitution $T-\tau-s=\xi$. According to mean value theorem, there exist the moments $-\tau\le s_1\le 0$, $T-\tau\le\xi_1\le T$, such that

$$B_n(T)=\frac{l}{k_n}\int_{T-\tau}^{T}\sin_\tau\{k_n,s\}\Phi_n''(T-\tau-s)ds\le\frac{l}{k_n}\tau\max_{-\tau\le s\le 0}|\Phi_n''(s_1)|\times$$

$$\times\max_{T-\tau\le\xi\le T}\left\{(\xi_1-\tau)-\left(\left(\frac{\pi n}{l}a\right)^2-c\right)^{\frac{3}{2}}\frac{\xi_1^3}{3!}+\ldots\right.$$

$$\ldots+(-1)^k\left(\left(\frac{\pi n}{l}a\right)^2-c\right)^{\frac{1}{2}(2k+1)}\frac{[\xi_1-(k-1)\tau]^{2k+1}}{(2k+1)!}\Bigg\}.$$

And if the conditions

$$\lim_{n\to\infty}\left|\Phi_n''(0)\right|n^{2k+3+\alpha}=0,$$

are fulfilled, then the series $S_2(x,t)$ converges absolutely and uniformly.

3. Consider the coefficients $C_n(t)$. For fixed moment of the time $(k-1)\tau\le T<k\tau$ make the substitution $T-\tau-\xi=s$. As it follows from the mean value theorem, there exists $T-\tau\le\xi_1\le T$, at which it holds the inequality

$$C_n(T)=\frac{l}{k_n}\int_{T-\tau}^{T}\sin_\tau\{k_n,\xi\}F_n(T-\tau-\xi)d\xi\le\left(\left(\frac{\pi n}{l}a\right)^2-1\right)^{-1}\tau\max_{-\tau\le s\le 0}\left|F_n(s)\right|\times$$

$$\times\max_{T-\tau\le\xi\le T}\Bigg\{(\xi_1-\tau)-\left(\left(\frac{\pi n}{l}a\right)^2-c\right)^{\frac{3}{2}}\frac{\xi_1^2}{3}+\ldots$$

$$\ldots+(-1)^k\left(\left(\frac{\pi n}{l}a\right)^2-c\right)^{\frac{1}{2}(2k+1)}\frac{[\xi_1-(k-1)\tau]^{2k+1}}{(2k+1)!}\Bigg\}.$$

And if the condition

$$\lim_{n\to\infty}\max_{T-\tau\le s\le T}\left|F_n(s)\right|n^{2k+3+\alpha}=0,$$

is fulfilled, then the series $S_3(x,t)$ converges absolutely and uniformly.

Thus, it is shown that for absolute and uniform convergence of the series $S_1(x,t)$, $S_2(x,t)$, $S_3(x,t)$ it is necessary "quick" decreases with respect to the index n of the Fourier's coefficients $\Phi_n''(t)$, $-\tau\le t\le 0$, $F_n(t)$, $T-\tau\le t\le T$. The convergence of derivatives of the solutions follows from the features of derivatives of the delayed sine and cosine represented by finite polynomials.

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GECİKMƏSİ OLAN RƏQS TƏNLİYİ HAQQINDA

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XÜLASƏ

Təqdim olunan işdə gecikməli kosinus və sinus adlanan xüsusi funksiyaların köməyilə "sırf gecikməsi" olan halda rəqslərin diferensial tənliyi üçün birinci sərhəd məsələsinin həlli sıra şəklində qurulmuşdur.

Açar sözlər: rəqs tənliyi, gecikmə, sərhəd məsələsi, Furye metodu, gecikməli kosinus, gecikməli sinus.

ОБ УРАВНЕНИИ КОЛЕБАНИЙ С ЗАПАЗДЫВАНИЕМ

Г.Ю.МЕХТИЕВА, Э.И. АЗИЗБЕКОВ, Д.Я.ХУСАИНОВ

РЕЗЮМЕ

В работе, используя специальные функции, называемые запаздывающими косинусом и синусом, построено решение первой краевой задачи для дифференциального уравнения колебаний «с чистым запаздыванием» в виде ряда.

Ключевые слова: уравнение колебания, запаздывание, краевая задача, метод Фурье, запаздывающий косинус, запаздывающий синус.

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